Characterization of Extinction Time and Approximation by Explicit Solutions for Total Variation Flow

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Objectives

We prove several new results and extensions of prior work on the Total Variation Flow (TVF) problem.

Introduction

The Total Variation Flow is classically given by the partial differential equation $u_t = \operatorname{div}_x \frac{\nabla_x u}{|\nabla_x u|}$. However, we cannot assume $|\nabla_x u| \neq 0$, so we instead consider the strong formulation

$$\begin{cases} u_t(x,t) = \operatorname{div}_x z(x,t) \text{ in } \mathcal{D}'(\mathbb{R}^N) \times (0,\infty) \\ |z| \leq 1 \\ \int_0^T \int_{\mathbb{R}^N} \nabla_x u \cdot z \, dx dt = \int_0^T \int_{\mathbb{R}^N} |\nabla_x u| \, dx dt. \end{cases} \quad (1)$$

We are interested in solutions to TVF due to its applications in image restoration and preventive policing, among others.

Approximations by Explicit Solutions

Theorem

Let $0 \leq u_I \in L^1_{loc}(\mathbb{R}^n)$ be radial initial data, u be the evolution of u_I under (1), and $f(|x|) := u_I(x)$. Then there exists a sequence of explicit solutions $(u_k(x,t))_k$ such that $u_k \xrightarrow{L^1_{loc}} u$ as $k \to \infty$.

Proposition

Let $0 \leq u_I : \mathbb{R}^n \to \mathbb{R}$ be a continuous compactly supported radial initial datum with profile curve f and let u be the evolution of u_I under (1). Then there exists an increasing sequence of explicit solutions $u_n(x,t)$ such $u_n \xrightarrow{L^1_{loc}} u$ as $n \to \infty$.

1D Case

Proposition (HKS 2016)

Let $-\infty < x_1 < \cdots < x_m < \infty$, $I_0 = (-\infty, x_1)$, $\forall i \in \{1,2,\cdots,m-1\}$ $I_i = [x_i, x_{i+1})$, and $I_m = [x_m, \infty)$. Let $A_0, \ldots, A_m \in \mathbb{R}$ and, $\forall i \in \{0,\cdots,m-1\}$, $A_i \neq A_{i+1}$ and $b_i = \operatorname{sgn}(A_{i+1} - A_i)$. The solution to (1) with initial datum $u_I = \sum_{i=0}^m A_i \chi_{I_i}$ is given by

$$u(x,t) = \sum_{i=1}^{m-1} \left(\frac{b_i - b_{i-1}}{x_i - x_{i-1}} t + A_i \right) \chi_{I_i}(x), \quad (2)$$

until the least time, $\hat{T}$, where $u(\cdot,\hat{T})|_{I_i} = u(\cdot,\hat{T})|_{I_{i+1}}$ for some i. We then consolidate the intervals and iterate the proposition using $u_I(x) = u(x,\hat{T})$. We continue in this fashion until the solution stabilizes.

Proposition (Regions of Constancy)

Let $u_I \in L^1_{loc}(\mathbb{R}^n)$ be radial, u be the solution emanating from u_I, and $J \subset \mathbb{R}^n$ be an annulus or ball centered at 0 such that $u_I(J) = \{c\}$ for some constant $c \in \mathbb{R}$. Then there is a continuous function, $c(t) : \mathbb{R} \to \mathbb{R}$, such that $u(J,t) = \{c(t)\}$.

Theorem (Local Behavior of Monotone Segments)

Let $u_I \in L^1_{loc}(\mathbb{R})$, $I \subset R$ be an open interval such that $u_I(I)$ has no local or global extrema, J be open so that $J \subset\subset I$, and u be the solution emanating from u_I. Then there is a $T \in \mathbb{R}$ with $(u|_{J\times(0,T)})(x,t) \equiv u_I(x)$.

Extinction Time

Proposition

Let $0 \leq u_I \in L^1_{loc}(\mathbb{R})$ and $u(x,t) : \mathbb{R} \times (0,T) \to \mathbb{R}$ be the evolution of u_I under (1). Then the extinction time of u is finite if and only if $u_I \in L^1$, in which case the following formula holds: $\int_{\mathbb{R}} u(x,t)\,dx = \int_{\mathbb{R}} u_I(x)\,dx - 2t$, $\forall t : 0 \leq t \leq T$.

Signed Data

Theorem

Suppose $u \in L^1_{loc}$ is continuous with compact support. Let x_n denote the finite values of x such that $u(x,0) = 0$ and $\exists \epsilon > 0$ such that for all $x_* \in (x_n - \epsilon, x_n)$, and $x^* \in (x_n, x_n + \epsilon)$, $u(x_*,0) \cdot u(x^*,0) < 0$ and points x_1, x_m such that $\forall x < x_1$, $u_I(x) = 0$ and $\forall x > x_m$, $u_I(x) = 0$. Next, let $R_n = [x_n, x_{n+1}]$. Finally, denote $M = \frac{1}{2}\max(\int_{R_1}|u_I|dx, \int_{R_2}|u_I|dx, \cdots, \int_{R_m}|u_I|dx)$. Then, the extinction time of u, T, is governed by the inequality

$$M \leq T \leq \frac{1}{2}\max\left(\int |u_I^+|\,dx, \int |u_I^-|\,dx\right).$$

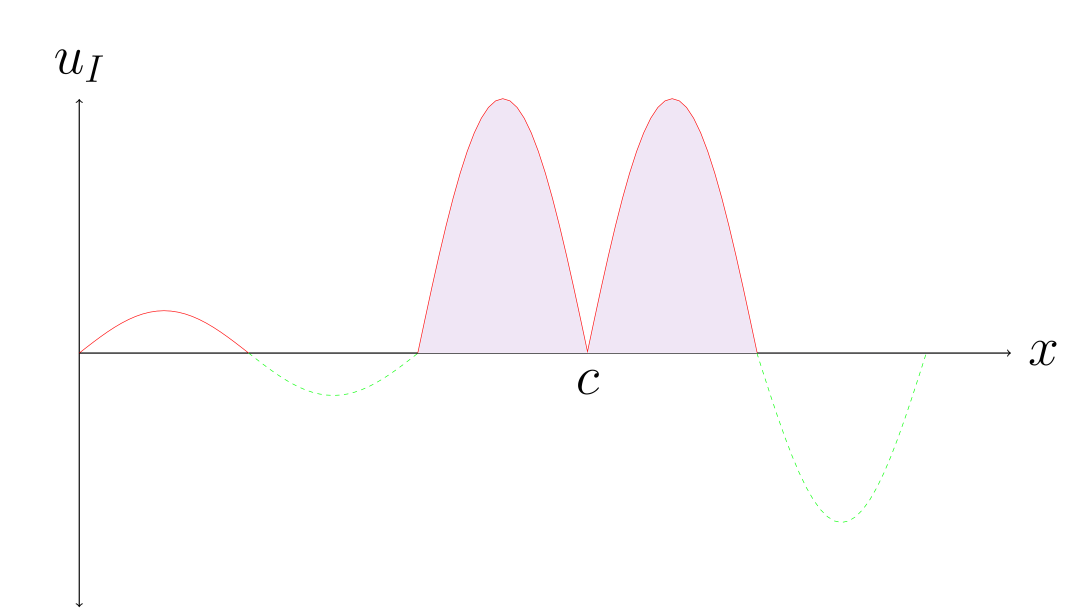

Figure 1: u_I split into u_I^+(red) and u_I^-(green) with the shaded region representing $2M$. Note $c \notin x_n$ as u_I does not change sign around c.

Barriers and Sheets

Proposition

Suppose that $u(x,t) : \mathbb{R}^N \times (0,\infty) \to \mathbb{R}$ is an entropy solution with initial datum $u_I \in L^1_{loc}(\mathbb{R}^N)$. Then for any $M \in \mathbb{N}$, $\tilde{u}(x,y,t) := u(x,t)$, $\tilde{u}$ is an entropy solution with initial datum $\tilde{u}_I \in L^1_{loc}(\mathbb{R}^N \times \mathbb{R}^M)$ and $\tilde{u}_I(x,y) := u_I(x)$.

References

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